PART I: Visual Arts & Mathematics

AGE RANGE: 13 – 15

TOOL 4: POLYHEDRA AND PERSPECTIVE

SPEL – Sociedade Promotora de Estabelecimentos de Ensino

TheArtOf Maths

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: Polyhedra and Perspective

Age range: 13 – 15 years old

Duration: 2 hours

Mathematical concepts: Polyhedra, Convex polyhedron, Platonic Solids, Hexahedron, Tetrahedron, Octahedron, Icosahedron, Dodecahedron

Artistic concepts: Linear perspective, Color perspective, Vanishing point

General objectives: Identify and recognize platonic solids; Analyse different techniques on how artists developed their skills throughout time, namely through the use of perspective.

Instructions and Methodologies: Have students see the paintings on big scale in order to get a clearer notion on perspective, either by projecting the paintings or by accessing the internet.

Resources: A pen, a ruler and colored pencils/crayons.

Tips for the educator: Prepare planifications of the platonic solids and hand them out to students in case they struggle to get across exercises 4 and 5. By being able to physically touch one they will understand better their concept.

Learning Outcomes and Competences: At the end of this tool, the student will be able to:

- To understand the logical process behind the different ways of how artists developed paintings through the use of linear and aerial perspective;
- To know the difference between linear and aerial perspective;
- To understand the concept behind a platonic solid and what takes a polyhedron to be considered one, as well as name them.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Polyhedra has had a relationship with Art for thousands of years.

Hundreds of artefacts resembling Polyhedra, believed to date from the Neolithic Period (around 5000.B.C), were found in Scotland. Some of them (Fig. 1) are now displayed in the Ashmolean Museum in Oxford.

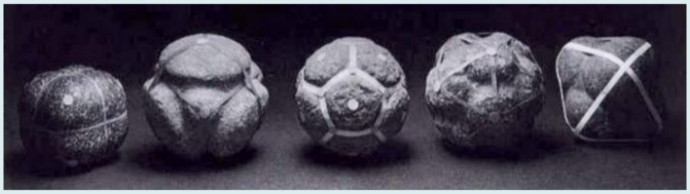

Fig. 1 – Stone-ball carved stones from the Neolithic Period
(Source: https://www.georgehart.com/virtual-polyhedra/neolithic.html)

In Ancient Greece, though, Polyhedra were a symbol of deep philosophical and religious truths. The mathematician and philosopher Plato (428 – 347 B.C.), in his written dialogue *Timaeus* (c.360 B.C.), went as far as to associate the five regular, convex Polyhedra, to the four basic elements believed to be the basis of the physical world – Air, Water, Earth and Fire (and the Universe).

As mythical as Plato's theory might have seemed, it influenced many other philosophers in the following centuries, so much to the point these solids became known as Platonic Solids. For instance, Johannes Kepler (1571-1630), when searching for a mathematical order in the world, inspired by this theory, represented the elements as seen in Fig. 2.

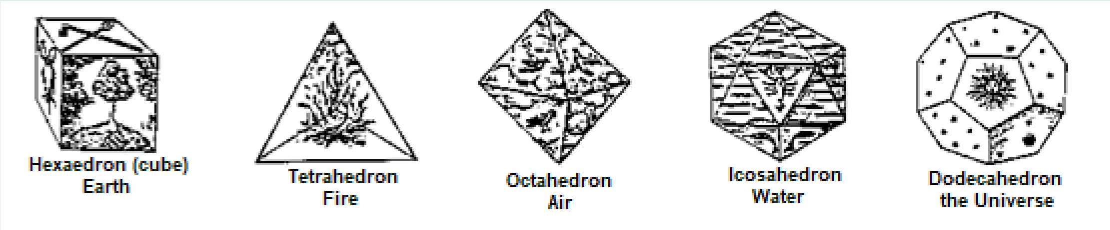

Fig. 2 – Plato's association of the Platonic Solids to the Elements, by Johannes Kepler.
(Source: http://thewondersofmathandart.blogspot.com/2012/11/patterns-in-void-platonic-solids-in.html)

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Polyhedra and Perspective

Linear Perspective

The relationship between Polyhedra and Art reached its peak during Renaissance (1300-1600), after the sculptor and architect Fillippo Brunelleschi (1377-1446) discovered the principles of linear perspective. It consisted in a technique that successfully created an illusion of depth on a flat surface.

After publishing his method, it spread throughout Italy and Europe, and other artists made use of it and perfected it even more. A great example of such implementation in paintings is the Leonardo da Vinci's masterpiece "Last Supper", as seen in Fig. 3.

Fig. 3 – Last Supper (1495-98), by Leonardo da Vinci
(Source: https://en.wikipedia.org/wiki/Last_Supper)

The concept of Polyhedra and Platonic Solids grew even broader as artists started making use of it in their compositions. These forms of illustration, using principles of linear perspective, made it explicit to distinguish between the front and back faces of the Polyhedron's polygons (Fig. 4), unlike earlier drawings in which the front and back polygons were visually undistinguishable and/or confusing.

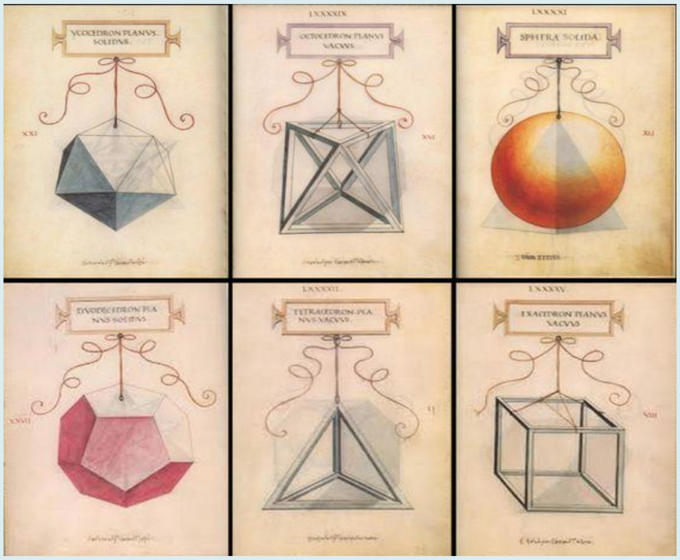

Fig. 4 – Polyhedra Illustrations for Luca Pacioli's book "The Divine Proportion" (1509), by Leonardo da Vinci. (Source: https://lifethroughamathematicianseyes.wordpress.com/2018/01/24/leonardo-da-vincis-geometric-sketches/)

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In modern times, the use of Polyhedra can been found in other forms of art, such as design and architecture. Check out Fig. 4, 5, 6 and 7 for some examples.

Fig. 5 - "Yin and Yang", by George W. Hart (Source: https://www.georgehart.com/sculpture/yin-yang.html)

Fig. 6 – Dome Auditorium of the La Seine Musicale, in Paris (Source: GraphyArchy [CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0)]

Fig. 7 – Habitable Polyhedron, by Manuel Villa (https://newatlas.com/habitable-polyhedron-pod/22858/#gallery

Fig. 8 – Building of the Casa da Música, in Porto (Source: https://commons.wikimedia.org/wiki/File:Casa-da-musica(exterior).1000.jpg#filelinks)

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Areal Perspective (color perspective)

After linear perspective was embraced and mastered, other ways to create an illusion of depth became a matter of study. One of them is known as aerial perspective, which lays on the rule of using darker colors and shades in the foreground of an illustration, and lighter colors in the background in a way to create depth.

A great example of aerial perspective use in paintings are those by the Romantic painter Caspar David Friedrich (1774-1840):

Fig. 9 – "Wanderer above the Sea of Fog" (1818), by Caspar David Friedrich (Source: https://www.wga.hu/frames-e.html?/html/f/friedric/2/209fried.html)

Fig. 10 – "Rocky Landscape in the Elbe Sandstone Mountains" (1822-23), by Caspar David Friedrich (Source: https://digital.belvedere.at/objects/8389/felsenlandschaft-imelbsandsteingebirge)

As it can be observed in Fig. 8 and 9, lighter and fuzzier colors represent a distant horizon/ background, whereas the darker colours, in the foreground, seem closer to the viewer. This use of colors in such a natural way affects the viewer's eyes to the point that it creates an illusion of depth.

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Glossary

Aerial Perspective (color perspective): A technique used in visual arts through the manipulation of color shades, in order to obtain a sense of depth in a 2-dimensional flat surface.

Linear Perspective: A technique used to make an approximate representation of a 3-dimensional image, as seen by the eye, on a 2-dimensional surface, by using parallel lines (orthogonals) intersecting into a single point of a horizon's composition.

Renaissance: Derived from the Italian word "Rinascimento". Renaissance represents a cultural rebirth, which marked a transition from the middle ages to modernity. It began in Florence, Italy, and spread throughout Europe.

Vanishing Point: A single point in a horizon's composition used in Linear Perspective, in which all parallel lines (orthogonals) converge into a single point. Usually, the protagonist of a composition lays in that point.

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The Math behind Perspective

Filippo Brunelleschi realized that if there were parallel lines converging into a single point in a canvas, it would create an illusion of depth. This fact led him to discovering a method in which, within a painting, he would be able to mathematically determine the right proportions of an object in scale to reality.

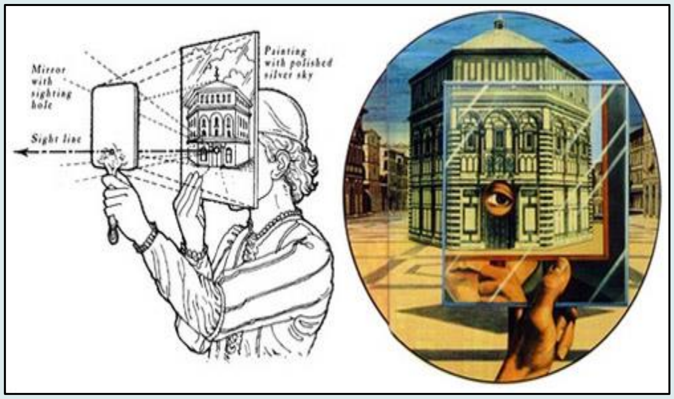

Fig. 11 – Experiment by Filippo Brunelleschi (Source: https://lifethroughamathematicianseyes.wordpress.com/2018/01/24/leonardo-da-vincis-geometric-sketches/)

In one of his most noted experiments, Filippo Brunelleschi sketched the Florence Baptistery on a canvas and made a single small hole in it. After that, he had a man holding it against the real Baptistery, with a mirror in between. He then asked the man what he was seeing. The man replied "oh, it is the Baptistery, Sir Filippo!". Brunelleschi had just depicted the building into perfect perspective.

To have a better understanding, take a look at the original painting of the "Last Supper" shown before (Fig. 3) and how it can be broken down into a linear perspective view in a successful attempt to create an illusion of depth (Fig. 12):

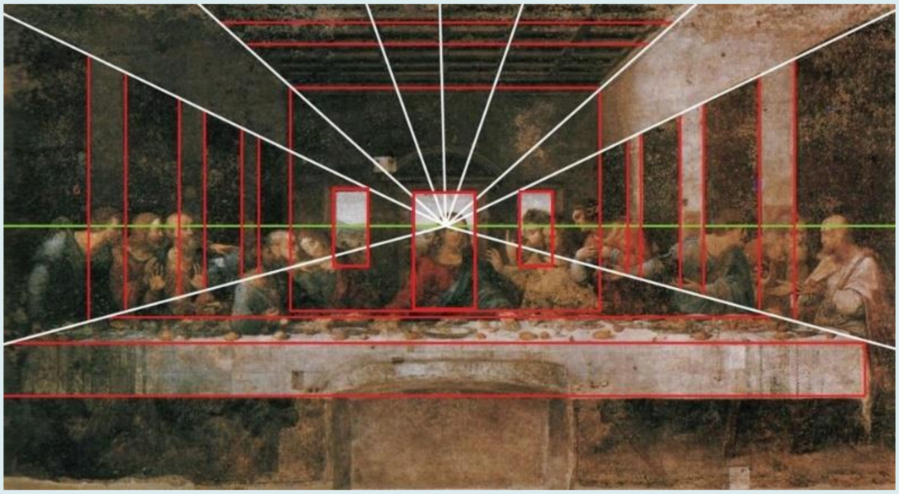

Fig. 12 – Presence of Linear Perspective in Last Supper (1495-98)
(Source: https://en.wikipedia.org/wiki/Last_Supper)

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As for the Aerial Perspective, one simply has to break the canvas down into “layers” and paint them accordingly.

A perfect illustration of this rule can be witnessed in the Figure 13: the darkest tones of painting are in the foreground. However, as the plains recede towards the horizon, the colors get paler, thus creating depth.

Fig. 13 – Aerial perspective representation
(Source: [Untitled], http://spartanartb.blogspot.com/2014/07/8th-grade-color-value-landscapes.html)

In the meantime, back to the Polyhedra, Leonhard Euler (1707-1783) had laid down the formula $\mathbf{V} - \mathbf{E} + \mathbf{F} = \mathbf{2}$, commonly known as Euler’s formula. This formula states that, in a convex Polyhedron, the number of vertices (V), minus the number of edges (E), plus the number of faces (F), always equals two.
For instance, as seen in Fig. 13, a hexahedron has 8 Vertices, 12 Edges and 6 Faces. This way, $\mathbf{V} - \mathbf{E} + \mathbf{F} = \mathbf{2}$ equals 8 – 12 + 6 = 2.

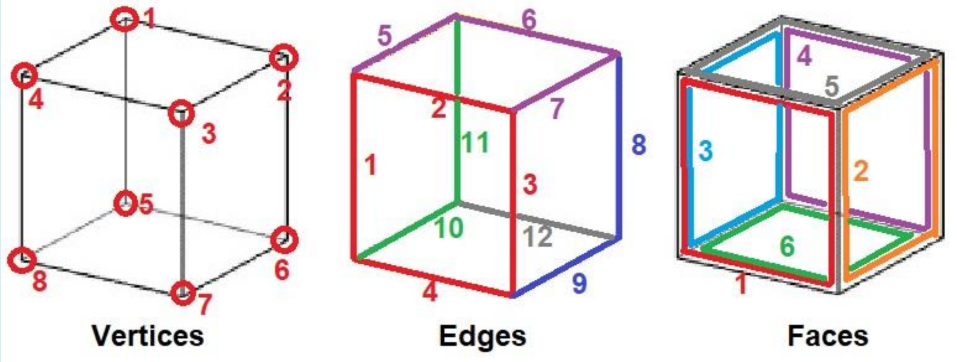

Fig. 14 – Vertices, Edges and Faces of a Hexahedron (Source: Author)

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TASKS

TASK 1

Given the principles of linear perspective:

1.1. Can you detect the vanishing point? Outline it in the black/white version.

Fig. 15 – Christ Handing the Keys to St. Peter (1481-82), by Pietro Perugino
(Source: https://commons.wikimedia.org/wiki/File:PeruginoKeys.jpg)

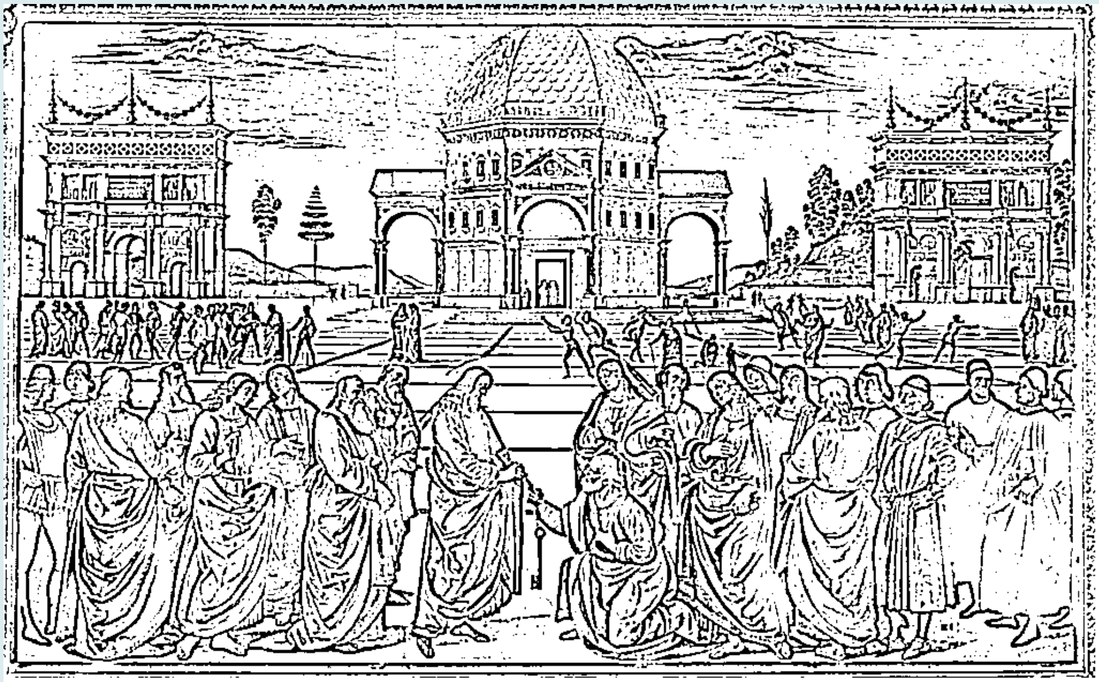

Fig. 16 – Christ Handing the Keys to St. Peter, by Pietro Perugino (1481-82)
(Source: https://commons.wikimedia.org/wiki/File:PeruginoKeys.jpg; Edited by: Author, 2019)

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1.2. Check out Fig. 17. Can you spot resemblances with the linear perspective principles in the following picture? Outline them in the black/white version (Fig. 18).

Fig. 17 – Rua 4, Espinho (2019) (Source: Author)

Fig. 18 – Rua 4, Espinho (2019) (Source: Author; Edited by: Author, 2019)

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1.3. Draw a horizon line and define your vanishing point. After that, draw three cubes, each one seen from a different perspective. (Tip: objects drawn above the horizon are seen as if you are looking under them, objects drawn below the horizon are seen as if you are looking over them)

TASK 2

Color the following landscape in a color perspective. (tip: Use darker colors in the foreground, mid-range tones in the center, and the background into a fading blue).

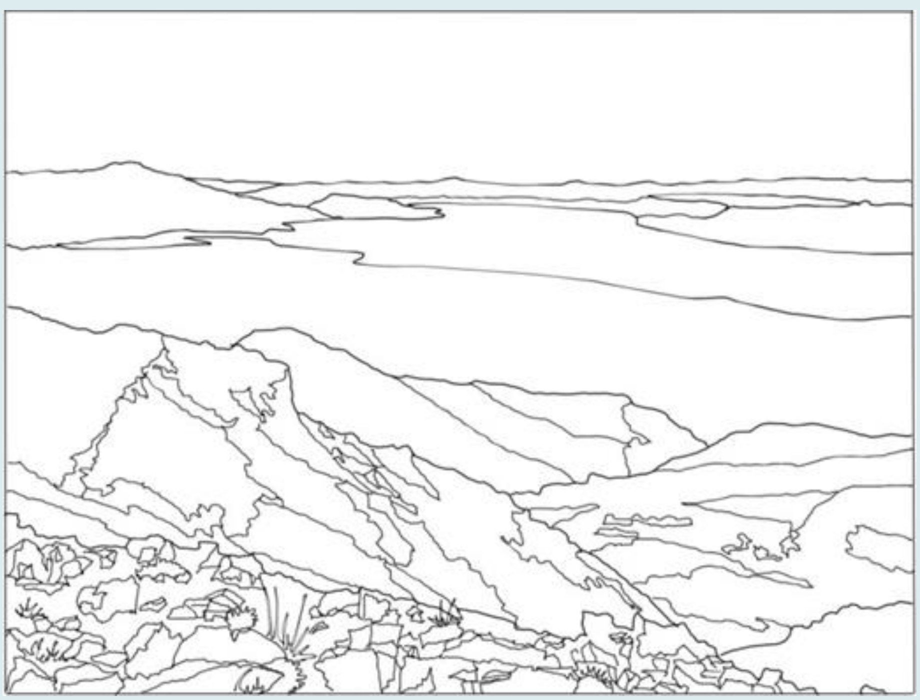

Fig. 19 – Canvas for aerial perspective training, by Erica Christensen
(Source: https://concepts.app/s/5aa9aa3c-529b-4875-91d3-e1a2366d299b)

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TASK 3

Perspective can be represented in illustrations through the use of Color or Linear Perspective.

Indicate, amongst the following paintings, which ones better represent Color Perspective, Linear Perspective, both, or none at all.

Fig. 20 – "Calling of the Apostles Peter and Andrew" (1370), by Lorenzo Veneziano, at Staatliche Museum. (Source: https://commons.wikimedia.org/wiki/File:15_Lorenzo_Veneziano,_Calling_of_the_Apostles_Peter_and_Andrew._1370_Staatliche_Museen,_Berlin..jpg)

Fig. 21 – "Mona Lisa" (1503), by Leonardo da Vinci (Source: https://pt.wikipedia.org/wiki/Mona_Lisa)

Fig. 22 – "The Healing of the Cripple and the Raising of Tabitha" (1424), by Masolino da Panicale
(Source:https://commons.wikimedia.org/wiki/File:Cappella_brancacci,_Guarigione_dello_storpio_e_resurrezione_di_Tabita_(restaurato),_Masolino.jpg)

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TASK 4

Given Euler's formula $V - E + F = 2$ (where V = Vertices, E = Edges and F = Faces), fill in the following table:

Platonic solid	Number of faces (F)	Number of vertices (V)	Number of edges (E)	E + 2	F + V
Hexahedron	6	8	12	14	14
Tetrahedron					
Octahedron					
Dodecahedron					
Icosahedron					

TASK 5

Behold the following Polyhedra planifications:

5.1 Which hexahedron correspond to each planification?

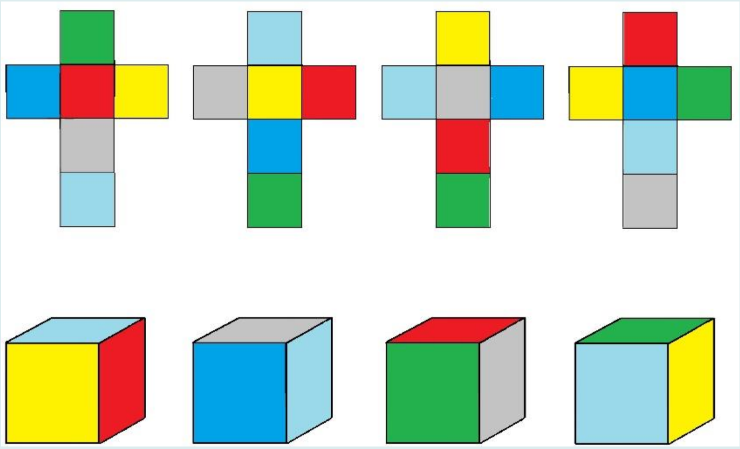

Fig. 23 – Planification of a Cube (Source: Author)

5.2 Circle the planifications that correspond to a platonic Polyhedron:

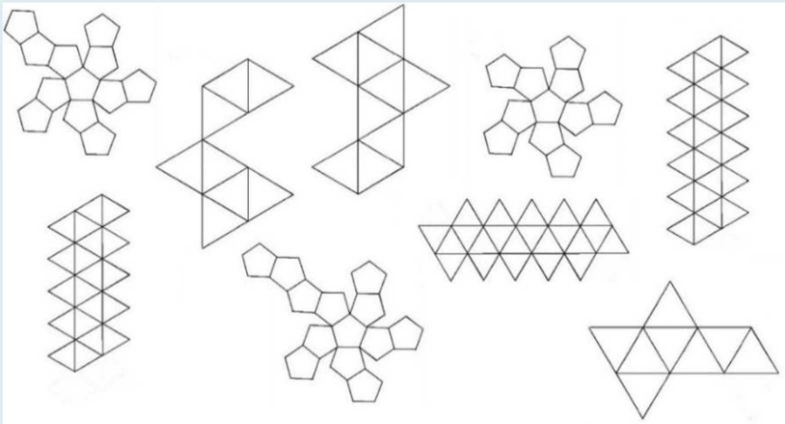

Fig. 24 – Planification of Polyhedra (Source: Author)

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LEARN MORE…

The History of Polyhedra in Greece

http://web.iyte.edu.tr/~gokhankiper/Polyhedra/Greeks.htm

Virtual Polyhedra, by George W. Hart

http://www.georgehart.com/virtual-polyhedra/vp.html

Interactive video on Linear Perspective

https://www.khanacademy.org/humanities/renaissance-reformation/early-renaissance1/beginners-renaissance-florence/a/linear-perspective-interactive

Presence of Linear Perspective in Paintings throughout the times

http://headforart.com/2016/07/01/linear-perspective/

Drawing Polyhedra in 1 point perspective

https://www.studentartguide.com/wp-content/uploads/2015/02/perspective-drawing.pdf

Leonardo da Vinci's Polyhedra

https://www.georgehart.com/virtual-polyhedra/leonardo.html

Color Perspective

https://www.exploring-landscape-painting.com/colour-perspective.html

Euler's Polyhedron formula, by Abigail Kirk

https://plus.maths.org/content/eulers-polyhedron-formula

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